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Superconformal harmonic surfaces in De Sitter space-times

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Dedicated to Professor Cristián U. Sánchez on the occasion of his 60th birthday.

Abstract

We consider harmonic maps of Riemann surfaces in De Sitter space-times $\mathbb{S}_1^n$, $n \geq 3$ with maximal isotropy dimension, also called superconformal. Harmonic sequences are constructed for these maps which are used to study their geometry. Global properties such as (linear) fullness and rigidity are discussed and polar maps of superconformal harmonic into odd-dimensional De Sitter space-times are studied. Lastly a characterization of superconformal minimal immersed tori is obtained generalizing a result by Sakaki [M. Sakaki, Space-like minimal surfaces in four-dimensional Lorentzian space forms, Tsukuba J. Math. 25 (2) (2001) 239–246].

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1. Introduction

In the last two decades there has been much progress in the study of harmonic maps from Riemann surfaces into compact symmetric spaces and Lie groups (see, for instance,

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[4,5] and their bibliography). From a pure differential–geometric viewpoint, the concept of harmonic map from a Riemann surface is a natural two-dimensional generalization of the concept of geodesic. In mathematical physics the interest in harmonic maps of Riemann surfaces stems from its relation with σ-models. A classical solution of a σ-model is a harmonic map or field into some (pseudo) Riemannian manifold. The study of σ-models on non-compact Lorentzian manifolds such as De Sitter space-times $\mathbb{S}_1^n$, and more generally on pseudo-Riemannian symmetric spaces and non-compact Lie groups arise in connection with problems of solid-state physics and has many other applications (see, for instance, [3] and the bibliography therein).

In this article we deal with superconformal harmonic maps $f : M \to \mathbb{S}_1^n$ of Riemann surfaces into De Sitter space-times $\mathbb{S}_1^n$ of dimension $n \geq 3$. We call a map $f : M \to \mathbb{S}_1^n$ superconformal if it has maximal isotropy dimension $r = \frac{n+1}{2} - 1$, where the notion of isotropy dimension is (mutatis–mutandis) the same as that introduced by Burstall [4] to study harmonic maps of Riemann surfaces into Euclidean spheres $\mathbb{S}^n$. In this way superconformal harmonic maps $f : M \to \mathbb{S}_1^n$ can be considered as natural generalizations of the so-called superconformal harmonic maps of Riemann surfaces into Euclidean spheres $\mathbb{S}^n$ introduced by Bolton et al. [9] (see also [8,20]).

For example, superconformal harmonic maps of surfaces in $\mathbb{S}_1^4$ arise as images of the conformal Gauss map of immersed Willmore surfaces in the Euclidean three sphere $\mathbb{S}^3$ and $\mathbb{R}^3$ as shown by Palmer [21]. Alias and Palmer [1] also considered superconformal minimal surfaces into four-dimensional Lorentz space forms and studied the behaviour of their normal and Gaussian curvature. Recently Sakaki [22] obtained a generalization of the so-called Ricci condition for superconformal minimal surfaces in four-dimensional Lorentz space forms. However a study of superconformal harmonic surfaces in higher-dimensional Lorentz space-forms seems still lacking.

Isotropic (i.e. infinite isotropy dimension) harmonic maps of Riemann surfaces into $\mathbb{S}_1^n$ were considered, for instance, by Ejiri [15] who proposed a *Bryant transform* to construct isotropic harmonic surfaces into $\mathbb{S}_1^n$ generalizing Bryant's method in [11]. Also a classification of harmonic isotropic maps of Riemann surfaces into $\mathbb{S}_1^n$ with non-degenerate osculating bundle was obtained by Erdem [16].

Our main goal here is to study geometric properties of superconformal harmonic maps $f : M \to \mathbb{S}_1^n$ by means of harmonic sequences which are constructed for these maps. This analytic tool allows us to study (linear) fullness properties, global rigidity and behaviour of normal curvatures. A construction of the so-called polar maps (or higher order Gauss transform) of superconformal harmonic maps into odd-dimensional targets $\mathbb{S}_1^{2m-1}$ is also considered. Finally we give a characterization of superconformal minimal immersion of tori in terms of the behaviour of the normal curvatures of the immersion, thus giving a generalization of a recent result by Sakaki [22].

The paper is organized as follows. In Section 2 we review some basic facts on the geometry of Lorentz manifolds. In Section 3 the notion of isotropy dimension due to Burstall [4] is applied to study harmonic maps of surfaces into $\mathbb{S}_1^n$, and a construction of harmonic sequences for maps with isotropy dimension greater than 1 is given. Section 4 deals with (linear) fullness properties of superconformal harmonic maps. In Section 5 we study the geometry of the normal bundles of a superconformal harmonic map f and use the information of the harmonic sequence determined by f to compute the normal curvatures and study their behaviour. Section 6 is devoted to study conditions under which two superconformal harmonic maps from a fixed Riemann surface are congruent. Polar maps of superconformal harmonic surfaces into odd-dimensional De Sitter space-times are defined and its main properties are considered in Section 7. Finally Section 8 deals with a characterization of superconformal minimal immersions of tori in terms of the normal curvatures of the immersion. This is one possible generalization of a result of Sakaki [22].

A twistorial construction and stability properties of superconformal and isotropic harmonic surfaces in $\mathbb{S}^n_1$ are considered in [18] and will appear elsewhere in due course.

2. Preliminaries

Let $\mathcal{L}^n$ be a Lorentz manifold with Lorentz metric $\langle , \rangle$. A map $f : M \to \mathcal{L}^n$ from a (pseudo)Riemannian manifold M is harmonic if it is an extreme of the energy functional $E(f, D) = \int_D \|\mathrm{d}f\|^2\,\mathrm{d}v$ for every compact subdomain $D \subseteq M$. It is shown that a map f is harmonic if and only if its tension field vanishes:

$$\tau(f) = 0 \tag{1}$$

This is the Euler–Lagrange system associated to the energy functional. It is a semi-linear (but not linear if $\mathcal{L}^n$ is not flat) system of partial differential equations which is elliptic if M is Riemannian (cf. [14]).

Now if M is a Riemann surface, then a map $f : M \to \mathcal{L}^n$ is called (weakly) conformal if for every local complex coordinate z on M,

$$\langle \partial f, \partial f \rangle^{\mathrm{c}} \equiv 0, \tag{2}$$

where $\langle\, , \,\rangle^{\mathrm{c}}$ denote the complex bilinear extension of the Lorentz metric, and $\partial = \frac{\partial}{\partial z}$. Call a map $f : M \to \mathcal{L}^n$ *space-like* if the pull-back $f^*\langle\, , \,\rangle$ of the ambient Lorentz metric is positive definite on the set of points $p \in M$ such that $\mathrm{d}f_p$ is non-singular. It easily follows that for $n \geq 3$ every (weakly) conformal map $f : M \to \mathcal{L}^n$ is space-like.

Note that if M is a orientable 2-manifold (i.e. a surface) and $f : M \to \mathcal{L}^n$ is an immersion (i.e. $\mathrm{d}f_p$ is non-singular for every $p \in M$) which is space-like then the pullback $f^*\langle\, , \,\rangle$ is a Riemannian metric on M which determines a Riemann surface structure on M such that f is a conformal immersion (see [19]). In this case if one considers on M the induced metric $g = f^*\langle\, , \,\rangle$ then $f : M \to \mathcal{L}^n$ is an isometric minimal space-like immersion. A conformal minimal immersion $f : M \to \mathcal{L}^n$ is also called *stationary* (cf. [2]).

Let $\mathbb{R}^{n+1}_1$ be the $(n+1)$-dimensional Minkowski space-time equipped with the indefinite non-degenerate Lorentz inner product of signature $(n, 1)$ given by

$$\langle x, y \rangle = x_0 y_0 + x_1 y_1 + \cdots + x_{n-1} y_{n-1} - x_n y_n \tag{3}$$

for $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n)$ in $\mathbb{R}^{n+1}_1$. According to relativity theory a nonzero $x \in \mathbb{R}^{n+1}_1$ is called *space-like* if $\langle x, x \rangle > 0$, *time-like* if $\langle x, x \rangle < 0$ and *light-like* or *null* if $\langle x, x \rangle = 0$. A subspace $V \subset \mathbb{R}^{n+1}_1$ is called space-like if every nonzero $v \in V$ is a space-like vector.

Let $H \subset \mathbb{R}^{n+1}_1$ be an hyperplane then it can be obtained as $H = \{x : \langle x, h\rangle = 0\}$ for some $h \in \mathbb{R}^{n+1}_1$. It is well known that the ambient Lorenz product $\langle\, ,\, \rangle$ induces on H an inner product which is positive definite if h is time-like, definite with signature $(n-1, 1)$ if h is space-like, and degenerate if h is light-like, respectively. Note that if h is light-like H contains h and H is the tangent space of the light cone $C = \{x : \langle x, x\rangle = 0\}$.

The (pseudo) Hermitian extension $\langle\, ,\, \rangle$ of the Lorentz inner product to $\mathbb{C}^{n+1}$ is given by

$$\langle z, w\rangle = z_0\bar{w}_0 + z_1\bar{w}_1 + \cdots + z_{n-1}\bar{w}_{n-1} - z_n\bar{w}_n. \tag{4}$$

Let $\mathbb{C}^{n+1}_1$ denote the complex vector space $\mathbb{C}^{n+1}$ equipped with the (pseudo-Hermitian) inner product (4). Then the complex bilinear extension of the Lorentz inner product is given by

$$\langle z, w\rangle^{\mathrm{c}} = \langle z, \bar{w}\rangle.$$

A complex vector subspace $W \subset \mathbb{C}^{n+1}_1$ will be called (complex) *isotropic* if $\langle z, w\rangle^{\mathrm{c}} = 0$ for any $z, w \in W$.

The real quadric $\mathbb{S}^n_1(R) = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = R^2\}$ is known as the De Sitter space-time, or pseudo-sphere of radius R and dimension n with constant sectional curvature $1/R^2$. The ambient Lorentz inner product (3) of $\mathbb{R}^{n+1}_1$ induces on $\mathbb{S}^n_1(R)$ a Lorentz metric of signature $(n-1, 1)$ denoted also by $\langle\, ,\, \rangle$. In this way $\mathbb{S}^n_1(R)$ becomes a Lorentz manifold on which the Lie Group $\mathrm{O}(n, 1)$ acts transitively by (pseudo) isometries.

Denote by $\mathbb{S}^n_1$ the De Sitter space-time of constant curvature one for short. It is easily seen that a map $f : M \to \mathbb{S}^n_1$ from a Riemann surface is harmonic if and only if it satisfies

$$\partial\bar{\partial} f = -\langle \partial f, \partial f\rangle f. \tag{5}$$

We recall here Ejiri's remarkable observation that as a consequence of Eq. (5), every harmonic map $f : M \to \mathbb{S}^n_1$ is real analytic with respect to complex local coordinates in M [15].

3. Superconformal harmonic maps

Given a harmonic map $f : M \to \mathbb{S}^n_1$ with $n \geq 2$ we follow Burstall [4] and define its *isotropy dimension* as the non-negative integer r for which the successive complex derivatives of f satisfy

$$\begin{aligned} &\langle \partial^\alpha f, \partial^\beta f\rangle^{\mathrm{c}} = 0 \quad \text{for } 1 \leq \alpha + \beta \leq 2r + 1, \text{ with } \alpha, \beta \geq 0, \\ &\langle \partial^{r+1} f, \partial^{r+1} f\rangle^{\mathrm{c}} \not\equiv 0. \end{aligned} \tag{6}$$

It is not hard to check that r so defined is independent of the choice of complex coordinates. Note that according to this definition a nowhere conformal map has isotropy dimension $r = 0$. On the contrary for $n \geq 3$ every harmonic map $f : M \to \mathbb{S}^n_1$ with isotropy dimension $r \geq 1$ is (weakly) conformal and hence a space-like map.

In order to study geometric properties of harmonic maps of surfaces into $\mathbb{S}^n_1$ we apply a standard Gram-Schmidt orthogonalization algorithm to the successive complex derivatives of a harmonic map $f : M \to \mathbb{S}^n_1$. This will give rise to the so-called *harmonic sequence* of f (cf. [7,8,10]).

Let $n \geq 3$ and $f : M \to \mathbb{S}_1^n$ a harmonic map of a Riemann surface. Fixed a complex chart $z : U \subset M \to \mathbb{C}$ for $j = 1, 2, \ldots$ define

$$\begin{aligned} f_0 &= f \\ f_{j+1} &= \partial f_j - \frac{\langle \partial f_j, f_j \rangle}{\| f_j \|^2} f_j. \end{aligned} \tag{7}$$

Condition $\langle f, f \rangle = 1$ implies

$$\langle \partial f, f \rangle = \langle \overline{\partial} f, f \rangle = 0 \tag{8}$$

so that $f_1 = \partial f$. Note that f_{j+1} is just the component of ∂f_j which is orthogonal to f_j so that $\langle f_j, f_{j+1} \rangle = 0$ for $j \geq 0$. It follows from (7) that $f_j(p) \in \text{span}_{\mathbb{C}}\{\partial f(p), \partial^2 f(p), \ldots, \partial^j f(p)\}$ for every point p in the domain of f_j. From now one we drop p and write simply

$$f_j \in \text{span}_{\mathbb{C}}\{\partial f, \partial^2 f, \ldots, \partial^j f\}.$$

Clearly f_{j+1} is defined in (7) away the zeros of $\| f_j \|^2$. These points coincide with the so-called *higher-order singularities of f* (cf. [6,7,12,24]). Note that in our situation the square norms $\| f_j \|^2$ might be negative or zero even if $f_j \neq 0$. The following Lemma establishes the orthogonality of the sequence $\{f_j\}$ and shows that the squared norms $\| f_j \|^2$ are positive open-densely.

Lemma 3.1. *Let M be a connected Riemann surface and $f : M \to \mathbb{S}_1^n$ for $n \geq 3$ a non-constant harmonic map with isotropy dimension $r \geq 1$. Fixed a complex chart (U, z) of M,* algorithm(7) *generates $\mathbb{C}_1^{n+1}$-valued maps $f_1, f_2, \ldots, f_r$ defined on an open and dense subset of U satisfying the following properties:*

(i) *For each $1 \leq j \leq r$ the zeros of f_j are isolated in U and $||f_j||^2 > 0$ on an open and dense subset of U.*
(ii) $\langle f_i, f_j \rangle = 0$ *for* $0 \leq i \neq j \leq r$.

Proof. We proceed by (finite) induction so that for $1 \leq k \leq r$ consider the statement $P(k)$: Algorithm (7) generates $\mathbb{C}_1^{n+1}$-valued maps $f_1, f_2, \ldots, f_k$ defined on an open dense subset of U satisfying the following properties:

(i) for each $1 \leq j \leq k$ the zeros of f_j are isolated in U and $\| f_j \|^2 > 0$ on an open and dense subset of U.
(i) $\langle f_i, f_j \rangle = 0$ for $0 \leq i \neq j \leq k$.

We prove first that $P(1)$ is true: If $f_1 = \partial f \equiv 0$ on an open subset $U' \subset U$, then

$$\partial^k f = \overline{\partial}^k f \equiv 0, \quad k = 1, 2, \ldots$$

on U'. Fixed a point $p \in U'$ one can assume that $z(p) = 0$ post-composing z with a translation if necessary. Now since f is real analytic we consider its Taylor expansion near p,

$$f(z,\bar z) = f(0,0) + \partial f(0,0)z + \bar\partial f(0,0)\bar z + \frac{1}{2!}(\partial^2 f(0,0)z^2 + 2\partial\bar\partial f(0,0)z\bar z + \bar\partial^2 f(0,0)\bar z^2) + \frac{1}{3!}(\partial^3 f(0,0)z^3 + 3\partial^2\bar\partial f(0,0)z^2\bar z + 3\partial\bar\partial^2 f(0,0)z\bar z^2 + \bar\partial^3 f(0,0)\bar z^3) + \cdots \tag{9}$$

Taking into account the harmonic map equation (5) we see that each term of the Taylor series above is zero except $f(0,0)$ and hence f is constant on the open set $U' \subset U$ and so it is constant on M. This shows that the zeros of f_1 must be isolated in U.

Let us now analyze the sign of $\|f_1\|^2$ on the open and dense subset $U'' \subset U$ on which $f_1 \neq 0$. Write $f_1 = A + \mathrm{i}B$ with A, B real vectors.[1] Then $\langle f_1, f_1\rangle^{\mathrm{c}} \equiv 0$ is equivalent to

$$\|A\|^2 = \|B\|^2, \quad \langle A, B\rangle = 0.$$

If $\|A\|^2(p) = \|B\|^2(p) < 0$ for some $p \in U''$ then $\{A(p), B(p)\}$ would span a real time-like 2-plane contained in $T_{f(p)}\mathbb{S}^n_1$ which is impossible. Therefore $\|f_1\|^2 \geq 0$ on U.

Note that if there were some point $p \in U''$ with $\|A(p)\|^2 = \|B(p)\|^2 = 0$ and $\{A(p), B(p)\}$ linearly independent, then $\{A(p), B(p)\}$ would span a real light-like 2-plane in $T_{f(p)}\mathbb{S}^n_1$ which is also impossible.

Now consider the (closed) set $\mathcal{O}$ of points $p \in U''$ such that $\|A(p)\|^2 = \|B(p)\|^2 = 0$ and $\{A(p), B(p)\}$ is a linearly dependent set. We claim that $\mathcal{O}$ consists of isolated points in U''. Assume that there is some open subset $V \subset U$ such that $V \subset \mathcal{O}$. Then we can write $f_1|_V = \gamma A$ with γ a complex non-vanishing function on V. Taking $\bar\partial$-derivative of f_1 using the harmonic map Eq. (5) and $\bar\partial A = \overline{\partial A}$ we get

$$\partial A \in \mathrm{span}_{\mathbb{C}}\{A\} \tag{10}$$

throughout V. Hence $\partial^j f \in \mathrm{span}_{\mathbb{C}}\{A\}$ on V for every $j \geq 1$. Consequently on V

$$\langle \partial^i f, \partial^j f\rangle^{\mathrm{c}} = 0, \quad 2 \leq i+j,\ i, j = 1, 2, \ldots$$

contradicting the fact that f has (finite) isotropy dimension $r \geq 1$. This proves our claim and so we conclude that $\|f_1\|^2 = 2\|A\|^2 = 2\|B\|^2 > 0$ must hold on an open and dense subset of U. We have proved part (i) of $P(1)$. Part (ii) is consequence of (8).

Now let $1 < k$ and assume that $P(k)$ is true. Then if $k = r$ the proof is complete. So let $k < r$ and note that from (7) f_{k+1} is defined on the domain of definition of f_k except possibly at isolated points at which $\|f_k\|^2 = 0$, hence on a dense open subset of U. We claim that the zeros of f_{k+1} are isolated in U. Suppose on the contrary that $f_{k+1} \equiv 0$ on an open subset $U' \subset U$. Then by definition of f_{k+1} this is equivalent to

$$\partial^{k+1} f \in \mathrm{span}_{\mathbb{C}}\{\partial f, \ldots, \partial^k f\}$$

on U'. Hence the following is true on U':

$$\partial^{k+s} f \in \mathrm{span}_{\mathbb{C}}\{\partial f, \ldots, \partial^k f\}\ \forall s \geq 1. \tag{11}$$

[1] $A = \frac{1}{2}\frac{\partial}{\partial x}$ and $B = -\frac{1}{2}\frac{\partial}{\partial y}$.

Now recall that f has isotropy dimension $r > k$ so that $\mathrm{span}_{\mathbb{C}}\{\partial f(p), \ldots, \partial^k f(p)\}$ is a (complex) isotropic subspace of $T^{\mathbb{C}}_{f(p)}\mathbb{S}^n_1$ for every $p \in U$. Therefore from (11) we get $\langle \partial^{k+s} f, \partial^{k+s} f\rangle^{\mathrm{c}} = 0$ on U' for every $s \geq 1$, which contradicts the fact that f has (finite) isotropy dimension $r \geq 1$.

Let us now analyze the sign of $\|f_{k+1}\|^2$ on the open and dense subset $U'' \subset U$ on which $f_{k+1} \neq 0$. Write $f_{k+1} = A + \mathrm{i}B$ with A and B real vectors and recall that since $k+1 \leq r$, f_{k+1} is (complex) isotropic: $\langle f_{k+1}, f_{k+1}\rangle^{\mathrm{c}} = 0$. This is equivalent to

$$\|A\|^2 = \|B\|^2, \qquad \langle A, B\rangle = 0.$$

Thus if $\|A(p)\|^2 = \|B(p)\|^2 < 0$ for some $p \in U''$, then $\{A(p), B(p)\}$ would be linearly independent spanning a real time-like 2-plane contained in $T_{f(p)}\mathbb{S}^n_1$ which is impossible. Hence $\|f_{k+1}\|^2 \geq$ on U''.

Note also that there is no $p \in U''$ for which $\|A(p)\|^2 = \|B(p)\|^2 = 0$ and such that $\{A(p), B(p)\}$ is linearly independent. Otherwise $\{A(p), B(p)\}$ would span a real light-like 2-plane in $T_{f(p)}\mathbb{S}^n_1$.

Now let $\mathcal{O}$ be the (closed) set of points $p \in U''$ satisfying $\|A(p)\|^2 = \|B(p)\|^2 = 0$ and $\{A(p), B(p)\}$ is linearly dependent. We claim that $\mathcal{O}$ consists of isolated points in U. For, assume that there is an open subset $V \subset U$ such that $V \subset \mathcal{O}$. Then we can write $f_{k+1}|_V = \gamma A$ with γ a complex non-vanishing function on V. This condition is equivalent to

$$\partial^{k+1} f = \gamma A + \sum_{j=1}^{k} a_j \partial^j f \tag{12}$$

on V, where $a_1, \ldots, a_k$ are non-identically vanishing complex functions. Taking the $\bar{\partial}$-derivative of $\partial^{k+1} f$ in (12) and using the harmonic map equation (5) on V, we conclude that

$$\bar{\partial} A \in \mathrm{span}_{\mathbb{C}}\{A, f, \partial f, \ldots, \partial^k f\}. \tag{13}$$

From (13) and $\bar{\partial} A = \overline{\partial A}$ we obtain

$$\partial A \in \mathrm{span}_{\mathbb{C}}\{A, f, \partial f, \bar{\partial} f, \ldots \partial^k f, \bar{\partial}^k f\}. \tag{14}$$

Finally using (14) and (12) once more we see that the following holds throughout V,

$$\partial^{k+s} f \in \mathrm{span}_{\mathbb{C}}\{A, f, \partial f, \bar{\partial} f, \ldots, \partial^k f, \bar{\partial}^k f\} \quad \forall s \geq 2 \tag{15}$$

Recall that $k+1 \leq r$ hence $r+1 = k+s$ for some $s \geq 2$ then we have

$$\langle \partial^{r+1} f, \bar{\partial}^j f\rangle = \langle \partial^{r+1} f, \partial^j f\rangle^{\mathrm{c}} = 0, \quad 0 \leq j \leq k$$

on V, since the isotropy dimension of f is $r \geq 1$ and $r+1 \leq r+1+j \leq 2r+1$. This says that $\partial^{r+1} f$ is orthogonal on V to the subspace $\mathrm{span}_{\mathbb{C}}\{f, \bar{\partial} f, \bar{\partial}^2 f, \ldots, \bar{\partial}^k f\}$. Then from (15) we conclude that on V

$$\partial^{r+1} f \in \mathrm{span}_{\mathbb{C}}\{A, \partial f, \ldots, \partial^k f\},$$

which clearly is an isotropic subspace. In particular $\langle \partial^{r+1} f, \partial^{r+1} f\rangle^{c} = 0$ contradicting the fact that f has isotropy dimension $r \geq 1$. This proves our claim.

We finally conclude that the last possibility must hold, namely $\| f_{k+1}\|^2 = 2\|A\|^2 = 2\|B\|^2 > 0$ on a dense and open subset of U. We have then proved part (i) of statement $P(k+1)$. To prove part (ii) of $P(k+1)$ note first that since $P(k)$ is true

$$\langle f_i, f_j\rangle = 0, \quad 0 \leq i \neq j \leq k.$$

Also from (7) we get

$$\langle f_j, f_{k+1}\rangle = \langle f_j, \partial f_k\rangle, \quad j = 0, \ldots, k-1,$$
$$\langle f_k, f_{k+1}\rangle = 0.$$

Computing $0 = \bar{\partial}\langle f_j, f_k\rangle = \langle \bar{\partial} f_j, f_k\rangle + \langle f_j, \partial f_k\rangle$ for $0 \leq j \leq k-1$ we obtain

$$\langle f_j, \partial f_k\rangle = -\langle \bar{\partial} f_j, f_k\rangle, \quad 0 \leq j \leq k-1. \tag{16}$$

On the other hand, using (7) and the orthogonality relations (ii) in statement $P(k)$ and taking into account that

$$0 = \bar{\partial}\langle f_j, f_{j-1}\rangle = \langle \bar{\partial} f_j, f_{j-1}\rangle + \langle f_j, \partial f_{j-1}\rangle.$$

it easily follows that

$$\bar{\partial} f_j = -\frac{\| f_j\|^2}{\| f_{j-1}\|^2} f_{j-1}, \quad 0 \leq j \leq k$$

Plugging this into (16) implies that $\langle f_j, f_{k+1}\rangle = 0$ for $0 \leq j \leq k$. This shows that $f_0, f_1, \ldots, f_{k+1}$ are mutually orthogonal thus proving that $P(k+1)$ is true. This completes the proof of the Lemma. □

Remark 3.1. As consequence of Lemma 3.1 for $n \geq 3$ every harmonic map $f : M \to \mathbb{S}^n_1$ with isotropy dimension $r \geq 1$ is space-like and $\mathrm{d} f_p$ is nonsingular for every point p in an open and dense subset of M. In particular $g = f^*\langle\, ,\, \rangle$ is defines a Riemannian metric (the induced metric) on M with isolated singularities. Considering M equipped with the induced metric g then $f : M \to \mathbb{S}^n_1$ is a branched[2] isometric minimal space-like immersion [24]. When $\mathrm{d} f_p$ is non-singular for every $p \in M$ the harmonic map f is called simply a minimal immersion. Such maps are also called stationary (cf. [2]).

Now observe that if the isotropy dimension of f is $r \geq 1$ the finite sequence $f_0, f_1, \ldots, f_r$ satisfies

$$\langle f_i, f_j\rangle^{c} = 0, \quad 1 \leq i, j \leq r. \tag{17}$$

[2] p is a branch point of f if $\mathrm{d} f_p = 0$.

Also by Lemma 3.1 we know that $f_0, f_1, \ldots, f_r$ satisfies the orthogonality relations

$$\langle f_i, f_j \rangle = 0, \quad 0 \le i \ne j \le r \tag{18}$$

Both conditions together imply that

$$\overline{f_r}, \overline{f_{r-1}}, \ldots, \overline{f_1}, f_0, f_1, f_2, \ldots, f_{r-1}, f_r$$

are mutually orthogonal therefore, $2r \le n$. Moreover using (7), orthogonal relations (18) and (17) it follows that f_{r+1} and $\overline{f_{r+1}}$ are both orthogonal to the complex subspace

$$\mathrm{span}_{\mathbb{C}}\{\overline{f_r}, \overline{f_{r-1}}, \ldots, \overline{f_1}, f_0, f_1, f_2, \ldots, f_{r-1}, f_r\}$$

from which $f_{r+1} \equiv 0$ if $2r = n$. In this case we shall say that *f* is *isotropic* (cf. [15]). It follows that $f : M \to \mathbb{S}^n_1$ is isotropic if and only if

$$\langle \partial^\alpha f, \partial^\beta f \rangle^{\mathrm{c}} = 0 \quad \forall \alpha, \beta \ge 0 \text{ such that } 1 \le \alpha + \beta$$

Such maps are also called *superminimal* or *pseudo-holomorphic* (cf. [4,20]). Well-known examples of isotropic maps are harmonic maps from the Riemann sphere into $\mathbb{S}^n_1$. For instance in [15] the author proves that every harmonic map $f : \mathbb{S}^2 \to \mathbb{S}^n_1$ is isotropic.

If $2r < n$ the orthogonality of f_{r+1} and $\overline{f_{r+1}}$ is measured by the complex function $\varphi_{r+1} =: \langle f_{r+1}, f_{r+1} \rangle^{\mathrm{c}} = \langle \partial^{r+1} f, \partial^{r+1} f \rangle^{\mathrm{c}}$ which does not vanish identically since *f* has isotropy dimension *r*. Note that as consequence of the harmonic map equation (5) φ_{r+1} is a holomorphic function and so its zeros are isolated.

In what follows for $n \ge 2$ put $n = 2m$ or $n = 2m - 1$, with $m \ge 1$. A harmonic map $f : M \to \mathbb{S}^n_1$ is called *superconformal* when it has isotropy dimension $r = m - 1$. Note that this terminology is rather confusing when applied to the extreme case of a map $f : M \to \mathbb{S}^2_1$, since it is superconformal if it is nowhere conformal. For any $m \ge 2$ superconformal harmonic maps into $\mathbb{S}^{2m}_1$ or $\mathbb{S}^{2m-1}_1$ are (weakly) conformal hence they are space-like maps. For *f* superconformal harmonic, $Q = \varphi_m \, \mathrm{d}z^{2m} = \langle f_m, f_m \rangle^{\mathrm{c}} \, \mathrm{d}z^{2m}$ is called the (holomorphic) $2m$th Hopf differential of *f*.

Note, for example, that a (weakly) conformal harmonic map $f : M \to \mathbb{S}^4_1$ is either isotropic or superconformal. Recall here that the four-dimensional De Sitter space $\mathbb{S}^4_1$ is recognized as the space of oriented 2-spheres in $\mathbb{S}^3$ and conformal maps $f : M \to \mathbb{S}^4_1$ are (conformal) Gauss maps of immersed surfaces in $\mathbb{S}^3$ (cf. [1,21]).

Let $f : M \to \mathbb{S}^3_1$ be a conformal minimal immersion. Fixed the induced metric $g = f^*\langle\, , \,\rangle$ on *M*, the second fundamental form of *f* is given by $\beta = -\langle \mathrm{d}f, d\mathbf{n} \rangle$, where $\mathbf{n}$ is a locally defined normal vector field along *f* with $\langle \mathbf{n}, \mathbf{n} \rangle = -\mathbf{1}$ (such $\mathbf{n}$ exist since *f* is space-like). On a local complex chart $(U, z = x + \mathrm{i}y)$ the matrix of β respect to the orthonormal basis $e_1 = \frac{1}{2\|f_1\|^2} \frac{\partial}{\partial x}$, $e_2 = \frac{1}{2\|f_1\|^2} \frac{\partial}{\partial y}$ is given by

$$B = \frac{1}{2\|f_1\|^2} \begin{pmatrix} \langle f_{xx}, \mathbf{n} \rangle & \langle f_{xy}, \mathbf{n} \rangle \\ \langle f_{xy}, \mathbf{n} \rangle & -\langle f_{xx}, \mathbf{n} \rangle \end{pmatrix}.$$

Let $\varphi_2\,dz^4$ be the 4th-Hopf differential of f. Then $\varphi_2 = \langle \partial^2 f, \partial^2 f\rangle^c = -\langle \partial^2 f, \mathbf{n}\rangle^{\mathbf{2}}$ and hence $|\varphi_2| = \frac{\|f_1\|^4}{8}D$, where $D = \frac{1}{\|f_1\|^4}(\langle f_{xx}, \mathbf{n}\rangle^{\mathbf{2}} + \langle \mathbf{f_{xy}}, \mathbf{n}\rangle^{\mathbf{2}})$ is the discriminant of the characteristic equation $\det(B - \lambda I) = 0$. A umbilic point of the immersed surface $f(M)$ is by definition a zero of D. Therefore a conformal minimal immersion $f : M \to \mathbb{S}^3_1$ is superconformal if and only if it has only isolated umbilic points.

In the following result, we summarize the main properties of the sequence $\{f_j\}$ generated by (7) from a superconformal harmonic map $f : M \to \mathbb{S}^n_1,\ n \geq 3$.

Corollary 3.2. *Let $f : M \to \mathbb{S}^n_1$ be a superconformal harmonic map, where $n = 2m$ or $n = 2m - 1$ and $m \geq 2$. Let $f_0, f_1, f_2, \dots, f_m$ be the finite sequence generated on a local complex chart (U, z) by (7). Defining*

$$f_{-j} := (-1)^j \frac{\overline{f_j}}{\|f_j\|^2}, \quad 1 \leq j \leq m, \tag{19}$$

the sequence $\{f_{-m}, \dots, f_{-1}, f_0, f_1, \dots, f_m\}$ satisfies

$$\begin{aligned} &f_{j+1} = \partial f_j - \partial \log \|f_j\|^2 f_j, \ -m \leq j \leq m-1,\\ &\bar\partial f_j = -\frac{\|f_j\|^2}{\|f_{j-1}\|^2} f_{j-1}, \qquad -m+1 \leq j \leq m, \end{aligned} \tag{20}$$

and the following orthogonality relations:

$$\begin{aligned} &\langle f_i, f_j\rangle = 0, \ \text{for}\ 0 < |i-j| \leq 2m-1,\\ &\langle f_m, f_{-m}\rangle = \frac{(-1)^m}{\|f_m\|^2}\varphi_m. \end{aligned} \tag{21}$$

Moreover, for $-(m-1) \leq j \leq m-1$, $\|f_j\|^2 > 0$ open densely on U.

Proof. (20) follows from (19) and (7) by straightforward computation. (21) is consequence of Lemma 3.1 and (19). □

3.1. Harmonic sequences

In order to study global properties of superconformal harmonic maps of surfaces in De Sitter space-times it is sometimes useful to extend the above construction to the whole of M. Let $\underline{\mathbb{C}}^{2m+1}_1 = \mathbb{C}^{2m+1}_1 \times M \to M$ be the trivial bundle endowed with the canonical connection $D_X s = Xs$ for any smooth local section s of $\underline{\mathbb{C}}^{2m+1}_1$ and $X \in TM$. The map f determines the complex line subbundle

$$L_0 = \{(v, x) \in \underline{\mathbb{C}}^{2m+1}_1 : v \in \mathbb{C} f(x)\}$$

equipped with the metric-compatible connection $\nabla_{L_0} = \pi_{L_0} \circ D$, where the projection $\pi_0 : \underline{\mathbb{C}}^{2m+1}_1 \to L_0$ along $L_0^\perp$ is well defined since $\langle f, f\rangle = 1$. By a well-knowntheorem of Koszul-Malgrange (see [14]), ∇_{L_0} determines a unique compatible holomorphic structure on L_0 such that a local smooth section s of $\underline{L_0}$ is holomorphic if and only if $\nabla''_{L_0}s = 0$ for any complex coordinate where $\nabla''_{L_0} = \pi_{L_0} \circ \bar{\partial}$. Hence s is holomorphic if and only if $\bar{\partial}s \in L_0^{\perp}$. In particular, the harmonic map equation (5) implies that f is a global holomorphic section of L_0. On the other hand, the fibers of L_0 determine a map $\varphi_0 : M \to \mathbb{CP}_1^{2m}$ by $\varphi_0(x) = \mathbb{C}f(x)$. Since φ_0 is the composition of f followed by the totally geodesic imbedding $\mathbb{S}_1^{2m} \hookrightarrow \mathbb{CP}_1^{2m}$, it results also harmonic.

Note that in general a complex vector subbundle $E \subset \underline{\mathbb{C}}_1^{n+1}$ can be equipped with the Koszul-Malgrange (see [14]) holomorphic structure provided it is non-degenerate respect to the ambient Hermitian indefinite inner product $\langle\ ,\ \rangle$. That is, $E \cap E^{\perp} = \{0\}$ fiberwise, where $\perp$ denotes $\langle\ ,\ \rangle$-orthogonal complement.

The bundle operator $A_{L_0} : TM \otimes L_0 \to L_0^{\perp}$ given by $A_{\underline{L_0}} = \pi_{L_0^{\perp}} \circ D$ splits up into its $(0, 1)$ and $(1, 0)$ parts $A'_{L_0} = \pi_{L_0^{\perp}} \circ \partial$ and $A''_{L_0} = \pi_{L_0^{\perp}} \circ \bar{\partial}$ according to the splitting $D = \partial + \bar{\partial}$. It is not hard to check that these operators are related by

$$(A'_{L_0})^* = -A''_{L_0^{\perp}}. \tag{22}$$

Now since f is a harmonic map, A'_{L_0} takes holomorphic sections of L_0 to holomorphic sections of $L_0^{\perp}$. This is equivalent to

$$A'_{L_0} \circ \nabla''_{L_0} = \nabla''_{L_0^{\perp}} \circ A'_{L_0},$$

which also says that A'_{L_0} is a holomorphic section of $\mathrm{Hom}(L_0, L_0^{\perp})$. Also by (22) A''_{L_0} is antiholomorphic.

Let L_1 be the unique complex line subbundle of $\underline{\mathbb{C}}_1^{2m+1}$ containing the image of A'_{L_0}. It is possible to define L_1 by continuity across the isolated zeros of A'_{L_0}, hence L_1 is a well-defined non-degenerate complex line subbundle of $\underline{\mathbb{C}}_1^{2m+1}$ on which the ambient metric $\langle\ ,\ \rangle$ is positive definite by Lemma 3.1. In particular it has a well-defined metric connection $\nabla_{L_1} = \pi_{L_1} \circ D$ and hence a unique compatible holomorphic structure. It is clear that A'_{L_0} sends holomorphic sections of L_0 to holomorphic sections of L_1, in particular from (7) it follows that $f_1 = A'_{L_0} f_0$ is a local holomorphic section of L_1. In the same way the image of the operator $A_{L_1} = \pi_{L_1^{\perp}} \circ D : L_1 \to L_1^{\perp}$ determines a unique non-degenerate complex line subbundle $L_2 \subset \underline{\mathbb{C}}_1^{2m+1}$, on which the ambient inner product is also positive definite. Thus it has also a well-defined metric connection $\nabla_{L_2} = \pi_{L_2} \circ D$ and hence a unique compatible holomorphic structure. Also from (7) $f_2 = A'_{L_1} f_1$ is a local holomorphic section of L_2. The process goes on producing a sequence of mutually orthogonal non-degenerate holomorphic complex line subbundles $L_1, L_2, \ldots, L_{m-1} \subseteq \underline{\mathbb{C}}_1^{2m+1}$ on which the ambient inner product is positive definite. Thus each has a well-defined metric connection $\nabla_{L_j} = \pi_{L_j} \circ D$ and a compatible holomorphic structure via the Koszul-Malgrange theorem. Unfortunately this is not the case with the last complex subbundle L_m containing the image of $A'_{L_{m-1}}$ which may degenerate at some points as occurs for example when $f : M \to \mathbb{S}_1^{2m}$ is full.

On the other hand, the ambient inner product is also positive definite on the conjugate bundles $L_{-j} = \overline{L}_j$ for $1 \leq j \leq m-1$. Including the possibly degenerate subbundles L_{-m}, L_m (on which the ambient inner product is not necessarily definite), Lemma 3.1 implies that the whole sequence $\{L_j : -m \leq j \leq m\}$ satisfies orthogonality relations

$$L_i \perp L_j \quad \text{for} \;\; 0 < |i-j| \leq 2m-1. \tag{23}$$

Also from (7) it follows that $A'_{L_j} : L_j \to L_{j+1}$ satisfies $A'_{L_j} f_j = f_{j+1}$. Further, $A'_{L_j} : L_j \to L_{j+1}$ is a holomorphic bundle operator[3] for $-m+1 \leq j \leq m-2$ since as consequence of (7) it satisfies

$$A'_{L_j} \circ \nabla''_{L_j} = \nabla''_{L_j^{\perp}} \circ A'_{L_j}.$$

In particular, the maps $\varphi_j : M \to \mathbb{CP}_1^{2m}$ given by $\varphi_j(x) = (L_j)_x$ are harmonic for $-m+1 \leq j \leq m-1$ (cf. [10]).

The finite sequence of harmonic maps $\varphi_j : M \to \mathbb{CP}_1^{2m}$, $-(m-1) \leq j \leq m-1$ is called the harmonic sequence of the initial superconformal harmonic map $f : M \to \mathbb{S}_1^{2m}$.

4. Linear fullness

According to Ejiri [15], a map f of a Riemann surface M to the De Sitter space $\mathbb{S}_1^n$ is said to be *full* if $f(M)$ is not contained in a *non-degenerate* hyperplane $H \subset \mathbb{R}_1^{n+1}$. However it might be contained in a degenerate hyperplane.

Theorem. [15] *Let $f : M \to \mathbb{S}_1^n$, $n \geq 3$ be a full isotropic harmonic map. Then, n is even* $(= 2m)$ *and $f(M)$ is contained in a unique degenerate hyperplane of* $\mathbb{R}_1^{2m+1}$.

It is then interesting to look for analogous results for superconformal harmonic maps into $\mathbb{S}_1^n$. To begin with let $f : M \to \mathbb{S}_1^{2m}$ be a superconformal harmonic map such that its image $f(M)$ is contained in a proper vector subspace V of $\mathbb{R}_1^{2m+1}$. Then for any local complex coordinate z of M the complex higher order derivatives $\overline{\partial}^{\alpha} f, \partial^{\beta} f$ all lie in $V^{\mathbb{C}}$ for all $\alpha, \beta \geq 0$ and so $L_j \subset \underline{V}^{\mathbb{C}}$ for $-m \leq j \leq m$. Consequently

$$\dim_{\mathbb{C}} \left[\oplus_{j=-m+1}^{m-1} L_j \right] + \dim_{\mathbb{C}}[L_m + \overline{L}_m] \leq \dim_{\mathbb{C}} \underline{V}^{\mathbb{C}} < 2m+1,$$

where $\underline{V}^{\mathbb{C}}$ is the trivial bundle $\underline{V}^{\mathbb{C}} = M \times V$. This inequality implies that $\dim_{\mathbb{C}}[L_m + \overline{L}_m] = 1$ (since f is superconformal), and $\dim_{\mathbb{C}} \underline{V}^{\mathbb{C}} = 2m$. Therefore $L_m = \overline{L}_m = L_{-m}$

[3] $A''_{L_j} : L_j \to L_{j-1}$ is a antiholomorphic bundle operator for $-m+2 \leq j \leq m-1$.

and V is a hyperplane in $\mathbb{R}_1^{2m+1}$. We have then the following orthogonal decomposition:

$$\underline{V}^{\mathbb{C}} = W \oplus L_m, \tag{24}$$

in which $W = \left[\oplus_{j=-m+1}^{m-1} L_j\right]$ is a complex $(2m-1)$-dimensional space-like subbundle of $\underline{V}^{\mathbb{C}}$.

An immediate application of the decomposition (24) is the following result which is the natural counterpart of Ejiri's theorem.

Theorem 4.1. *The image $f(M)$ of a non (linearly) full superconformal harmonic map $f : M \to \mathbb{S}_1^{2m}$ cannot be contained in any degenerate hyperplane of $\mathbb{R}_1^{2m+1}$.*

Proof. If the image $f(M)$ were contained in a degenerate hyperplane V, then there would be a nonzero light-like vector $\mathbf{n} \in \mathbb{R}_1^{2m+1}$ such that $V = \mathbf{n}^{\perp}$. Then $\mathbf{n} \in V$ and according to decomposition (24), the vector $\mathbf{n}$ would be contained in L_m since its orthogonal projection onto W is zero. Then $L_m = \mathbb{C}\mathbf{n}$ and so $\varphi_m \equiv 0$ which is a contradiction. □

Remark 4.1. The above result implies in particular that Ejiri's notion of fullness coincides for superconformal harmonic maps with linear fullness in the usual sense, i.e. not having image in any proper vector subspace.

Now if $f : M \to \mathbb{S}_1^{2m}$ is (linearly) full, $f(M)$ is contained in no proper vector subspace of $\mathbb{R}_1^{2m+1}$ and hence we can decompose

$$\underline{\mathbb{C}}_1^{2m+1} = W \overset{\perp}{\oplus} [L_m + \overline{L}_m] \tag{25}$$

from which it follows that $\dim_{\mathbb{C}}[L_m + \overline{L}_m] = 2$ hence $L_m \neq \overline{L}_m$ and so $L_m \oplus \overline{L}_m$ is a two-dimensional non-degenerate complex bundle. Note that W is a maximal complex space-like bundle and hence the ambient pseudo-Hermitian metric of $\underline{\mathbb{C}}_1^{2m+1}$ induces on $L_m \oplus \overline{L}_m$ a pseudo-Hermitian metric of signature (1, 1) so that $L_m \oplus \overline{L}_m$ is isometric to $\underline{\mathbb{C}}_1^2$.

On the other hand, if f is not full, then $f(M)$ is contained in a non-degenerate hyperplane V and not in a proper subspace of V by (24). Then either V is space-like and hence the ambient metric is positive definite on L_m, or V has signature $(2m-1, 1)$ and the ambient metric is negative definite on L_m.

Proposition 4.2. *Let $f : M \to \mathbb{S}_1^{2m}$ be a superconformal harmonic map. Then for every local complex chart on M the following inequality holds*:

$$|\,\|f_m\|^2| \leq |\varphi_m|. \tag{26}$$

If f is not full then its image f(M) lies fully in a non-degenerate hyperplane $V \subset \mathbb{R}_1^{2m+1}$ and equality holds in (26) for every local complex chart on M.

Proof. If f is full, let $X =: \overline{f_m} - \frac{\overline{\varphi_m}}{\|f_m\|^2} f_m$. Then X is a non-zero section of $L_m \oplus \overline{L}_m$ defined away the zeros of $\|f_m\|^2$ and satisfies $\langle X, f_m \rangle = 0$. Since $L_m \oplus \overline{L}_m$ is isometric to $\underline{\mathbb{C}}^2_1$, then $\|f_m\|^2 > 0$ if and only if $\|X\|^2 = \|f_m\|^2 - \frac{|\varphi_m|^2}{\|f_m\|^2} < 0$, from which we get $\|f_m\|^2 < |\varphi_m|$. Also $\|f_m\|^2 < 0$ if and only if $\|X\|^2 > 0$ and so $\|f_m\|^4 < |\varphi_m|^2$, which implies $\|f_m\|^2 > -|\varphi_m|$. We conclude that $|\,\|f_m\|^2| < |\varphi_m|$.

In the non-full case $\dim_{\mathbb{C}}[L_m + \overline{L}_m] = 1$, hence $L_m = \overline{L}_m$. Then for any local complex coordinate on M we have $\overline{f_m} = \lambda f_m$ for some local complex function λ. Thus $|\lambda| = 1$ and $\varphi_m = \overline{\lambda}\|f_m\|^2$. So that $|\varphi_m|^2 = \|f_m\|^4$ from which we get $\|f_m\|^2 = \pm|\varphi_m|$. Combining these two cases we obtain (26). $\square$

In the next section, we shall be able to prove the converse: if equality holds on an open subset (clearly equality holds at the isolated zeros of φ_m) then f cannot be linearly full.

Remark 4.2. From the proof of (26) above it follows that $\overline{L}_m = L_m$ and hence the sequence L_j generated by a non-full superconformal harmonic map $f : M \to \mathbb{S}^{2m}_1$ is $2m$-periodic:

$$L_{2m+j} = L_j, \quad j \in \mathbb{Z}.$$

An easy consequence of this fact is that every superconformal harmonic map f with target an odd-dimensional De Sitter space-time $\mathbb{S}^{2m-1}_1$ is (linearly) full.

5. Normal curvatures

Here we use the information contained in the harmonic sequence of a superconformal harmonic map $f : M \to \mathbb{S}^{2m}_1$ to study the behaviour of the Gaussian and normal curvatures of f. We will recognize these invariants to be the curvatures of the complex line bundles L_j determined by f.

Throughout we shall make use of some notions and facts which hold for harmonic maps of surfaces into Riemannian n-space forms, details of which may be found, for instance, in [6,8]. However, to keep the paper at a reasonable length, we shall omit the proofs since thanks to Lemma 3.1, the relevant parts of that discussion carry over to the case of superconformal harmonic maps $f : M \to \mathbb{S}^n_1$.

Let $f : M \to \mathbb{S}^{2m}_1$, $m \geq 2$ be a superconformal harmonic map. Fix the induced metric g on M, then f is a (branched) space-like harmonic (or minimal) isometric immersion (cf. Remark 3). Therefore the computations that follows can be carried out away the isolated singularities of the induced metric g. Let ∇ be the pseudo-Riemannian Levi–Civita connection of $\mathbb{S}^{2m}_1$ determined by the Lorentz metric and consider the pull-back bundle

$$T = f^*(T\mathbb{S}^{2m}_1) \subset \underline{\mathbb{R}}^{2m+1}_1 = \mathbb{R}^{2m+1}_1 \times M$$

with the pull-back connection denoted also by ∇ and the pull-back Lorentz metric $\langle\, ,\, \rangle$. The subspace of T_p generated by the ∇-derivatives of f up to order j is called the jth osculating space at $p \in M$ and is denoted by T^j_p (note that $T^1_p = \mathrm{d}f_p(TM)$). Clearly T^j_p is a subspace of T^{j+1}_p and the orthogonal complement of T^j_p in T^{j+1}_p, denoted by N^j_p is called the j-normal space at p. Then

$$T_p^j = T_p^{j-1} \oplus N_p^{j-1}, \quad 2 \le j \le m. \tag{27}$$

At "generic" points one can consider the jth osculating bundle T^j with $2j$-dimensional fibers T_p^j and also the jth normal bundle N^j with two-dimensional fibers N_p^j. A point p is called generic if the fiber of T^j over p coincides with the jth osculating space at p. It is shown that the set of generic points is open and dense in M. The set of non-generic points, also called higher-order singularities of f, consists of isolated points (cf. [12,23]).

For $1 \le j \le m-1$ the fibers of each complex line bundle L_j determined by f are isotropic complex lines in $\mathbb{C}_1^{2m+1}$ on which the ambient pseudo-Hermitian metric is positive definite by Lemma 3.1. Hence each L_j, $1 \le j \le m-1$, may be identified with an oriented real space-like 2-plane subbundle of $\mathbb{R}_1^{2m+1}$ in the following way: L_j has a local holomorphic section f_j generated on a local complex chart (U, z) by (7). Define real vector fields F_{2j-1}, F_{2j} on U such that

$$f_j = \frac{\|f_j\|}{\sqrt{2}}(F_{2j-1} - \mathrm{i}F_{2j}) \tag{28}$$

Since $\langle f_j, f_j\rangle^{\mathrm{c}} = 0$, the fields are orthogonal $\langle F_{2j-1}, F_{2j}\rangle = 0$ and of unit norm $\|F_{2j-1}\|^2 = \|F_{2j}\|^2 = 1$. Thus for $j = 1$, F_1, F_2 are local generating sections of the first osculating bundle or tangent bundle $T^1 = \mathrm{d}f(TM)$ of f, and for $2 \le j \le m-1$, F_{2j-1}, F_{2j} are local generating sections of the $(j-1)$th normal bundle N^{j-1} of f [6,8,12,23]. This establishes the identification of L_1 with $df(TM)$ and of L_j with N^{j-1} for $2 \le j \le m-1$.

By the above identifications the orthogonal direct sum (complex) maximal isotropic space-like subbundle

$$L_1 \oplus L_2 \oplus \cdots \oplus L_{m-1} \subset T^{\mathbb{C}}$$

identifies with $(m-1)$th osculating bundle $T^{m-1} \subset T$ of f. Note also from (28) that

$$\mathrm{d}f(TM)^{\mathbb{C}} = \bar{L}_1 \oplus L_1, \qquad (N^{j-1})^{\mathbb{C}} = \bar{L}_j \oplus L_j, \quad 2 \le j \le m-1.$$

It follows from Lemma 3.1 and our discussion above that T^{m-1} is a real space-like (hence non-degenerate) $2(m-1)$-dimensional vector subbundle of T. Now if f is full, we have $T^m = T$ and by (27) the last normal bundle $N^{m-1} = (T^{m-1})^{\perp}$ of f is a real non-degenerate oriented Lorentz 2-plane subbundle of T. That is, the restriction of the Lorentz metric to the fibers of N^{m-1} has signature (1, 1) and so N^{m-1} is isometric to $\mathbb{R}_1^2$. Then there are local generating sections F_{2m-1}, F_{2m} of N^{m-1} on U satisfying

$$\langle F_{2m-1}, F_{2m}\rangle = 0, \qquad \|F_{2m-1}\|^2 = -\|F_{2m}\|^2 = 1. \tag{29}$$

In particular $(N^{m-1})^{\mathbb{C}} = \bar{L}_m \oplus L_m$ and hence there are (local) functions α, β such that

$$f_m = \alpha F_{2m-1} - \beta F_{2m}, \tag{30}$$

so that L_m identifies with N^{m-1}.

Note that the direct sum subbundle $L_2 \oplus L_3 \oplus \cdots \oplus L_m$ identifies with the normal bundle

$$\nu(f) = N^1 \oplus N^2 \oplus \cdots \oplus N^{m-1}$$

of f and ∇ restricted to $\nu(f)$ coincides with the normal connection $\nabla^{\perp}$ on $\nu(f)$. Also ∇ restricted to T^1 coincides with the Levi–Civita Riemannian connection on M determined by the induced metric g. The projection of $\nabla^{\perp}$ onto each normal 2-plane subbundle N^{j-1} defines a metric-compatible connection $\nabla^{\perp}_{j-1}$, $2 \le j \le m$. Hence $\nabla^{\perp}_{j-1}$ is Riemannian for $2 \le j \le m-1$ whereas $\nabla^{\perp}_{m-1}$ is pseudo-Riemannian or Lorentzian.

Let $\omega_j = \langle \nabla^{\perp}_{j-1} F_{2j}, F_{2j-1} \rangle$ be the connection forms of T^1 ($j = 1$) and N^{j-1} (for $2 \le j \le m-1$). Then the respective curvature functions are given by $\mathrm{d}\omega_j = K_j\,\mathrm{d}A$ where $\mathrm{d}A = 2\|f_1\|^2\,\mathrm{d}x \wedge \mathrm{d}y$ is the area element of the induced metric g respect to a local complex coordinate $z = x + \mathrm{i}y$ (cf. [23]). If we let $\sigma_j =: \langle \partial F_{2j}, F_{2j-1} \rangle$, then straightforward computation shows that $\omega_j = 2\,\mathrm{Re}(\sigma_j\,\mathrm{d}z)$ from which $\mathrm{d}\omega_j = -4\,\mathrm{Im}(\bar{\partial}\sigma_j)\,\mathrm{d}x \wedge \mathrm{d}y$ follows. Both expressions together yield

$$K_j = -\frac{2}{\|f_1\|^2}\mathrm{Im}(\bar{\partial}\sigma_j), \quad 1 \le j \le m-1. \tag{31}$$

Now from $\partial\langle F_{2j}, f_j \rangle = \langle \partial F_{2j}, f_j \rangle$ and (28), we obtain $\mathrm{i}\partial\|f_j\| = \|f_j\|\sigma_j$ or equivalently,

$$\partial \log \|f_j\|^2 = -2\mathrm{i}\sigma_j.$$

Taking $\bar{\partial}$-derivative of this last expression and using (31) we get

$$K_j = -\frac{1}{2}\Delta_g \log \|f_j\|^2 \quad \text{for } j = 1, \ldots, m-1, \tag{32}$$

where $\Delta_g = 2\|f_1\|^{-2}\partial\bar{\partial}$ is the Laplacian of the induced metric $g = 2\|f_1\|^2\,\mathrm{d}z\,\mathrm{d}\bar{z}$ on M. Note in particular that K_1 is just the Gauss curvature of the induced metric g on M.

Remark 5.1. As shown in [6] it follows that K_j is the curvature of L_j respect to its metric-compatible connection ∇_{L_j} defined in the previous sections.

The computation of the last normal curvature K_m is slightly more involved than that of $K_1, \ldots, K_{m-1}$. K_m is defined as in the previous cases by $\mathrm{d}\omega_m = K_m\,\mathrm{d}A$ where $\omega_m = \langle \nabla^{\perp}_{m-1} F_{2m}, F_{2m-1} \rangle$ is the curvature form of the last normal bundle N^{m-1}, and $\mathrm{d}A$ is the area element of the induced metric g. Defining $\sigma_m = \langle \partial F_{2m}, F_{2m-1} \rangle$, then respect to a complex coordinate $z = x + \mathrm{i}y$ one obtains $\mathrm{d}\omega_m = -4\,\mathrm{Im}(\bar{\partial}\sigma_m)\,\mathrm{d}x \wedge \mathrm{d}y$. Then

$$K_m = -2\|f_1\|^{-2}\,\mathrm{Im}(\bar{\partial}\sigma_m). \tag{33}$$

Another expression for K_m can be obtained by computing $\bar{\partial}\sigma_m$. First we need the following result.

Lemma 5.1. $\partial f_m \in \overline{L_m} \oplus L_m \oplus L_{-m+1}$.

Proof. Since $\langle f_m, f_j\rangle = 0$ for $-m+1 \le j \le m-1$ then

$$0 = \partial\langle f_m, f_j\rangle = \langle \partial f_m, f_j\rangle + \langle f_m, \bar{\partial} f_j\rangle = \langle \partial f_m, f_j\rangle - \frac{\| f_j\|^2}{\| f_{j-1}\|^2}\langle f_m, f_{j-1}\rangle.$$

Hence $\langle \partial f_m, f_j\rangle = \frac{\| f_j\|^2}{\| f_{j-1}\|^2}\langle f_m, f_{j-1}\rangle$ and consequently $\langle \partial f_m, f_j\rangle = 0$ for $-m+2 \le j \le m-1$. □

Now the projection of ∂F_{2m} onto $L_{-m+1} = \overline{L}_{m-1}$ can be obtained as follows:

$$0 = \partial\langle F_{2m}, \overline{f_{m-1}}\rangle = \langle \partial F_{2m}, \overline{f_{m-1}}\rangle + \langle F_{2m}, \overline{\partial f_{m-1}}\rangle$$

Since $\partial f_{m-1} = f_m + \partial \log \| f_{m-1}\|^2 \cdot f_{m-1}$, we have $\langle \partial F_{2m}, \overline{f_{m-1}}\rangle = -\langle F_{2m}, \overline{f_m}\rangle$ and so the projection of ∂F_{2m} onto L_{-m+1} is given by $-\beta\| f_{m-1}\|^{-2} \cdot \overline{f_{m-1}}$. In the same way we deduce that the projection of ∂F_{2m-1} onto L_{-m+1} is given by $-\alpha\| f_{m-1}\|^{-2} \cdot \overline{f_{m-1}}$. Taking ∂-derivative in (30) we apply Lemma 5.1 to conclude that $\partial F_{2m}, \partial F_{2m-1} \in \overline{L_m} \oplus L_m \oplus L_{-m+1}$. Summing up we obtain

$$\begin{aligned} \partial F_{2m-1} &= \sigma_m F_{2m} - \alpha\| f_{m-1}\|^{-2} \cdot \overline{f_{m-1}}, \\ \partial F_{2m} &= \sigma_m F_{2m-1} - \beta\| f_{m-1}\|^{-2} \cdot \overline{f_{m-1}}. \end{aligned} \tag{34}$$

Plugging these two equations into

$$\bar{\partial}\sigma_m = \langle \bar{\partial}\partial F_{2m}, F_{2m-1}\rangle + \langle \partial F_{2m}, \partial F_{2m-1}\rangle, \tag{35}$$

we obtain

$$\mathrm{Im}(\bar{\partial}\sigma_m) = -\mathrm{Im}(\alpha.\bar{\beta})\| f_{m-1}\|^{-2},$$

which inserted in (33) produces the following formula for the last normal curvature:

$$K_m = 2\| f_1\|^{-2}\| f_{m-1}\|^{-2}\,\mathrm{Im}(\alpha\bar{\beta}). \tag{36}$$

Remark 5.2. When $f : M \to \mathbb{S}_1^{2m}$ is not full, it follows from the proof of Theorem 5.4 in the next section, that $K_m \equiv 0$ and $\dim N^{m-1} = 1$.

Proposition 5.2. *The Gaussian and normal curvatures $K_1, K_2, \ldots, K_m$ of a superconformal map f satisfy the following identities:*

$$\sum_{j=1}^{m-1} K_j - 1 = -\| f_1\|^{-2}\| f_{m-1}\|^{-2}(|\alpha|^2 - |\beta|^2), \tag{37}$$

$$\left(1 - \sum_{j=1}^{m-1} K_j\right)^2 + K_m^2 = \| f_1\|^{-4}\| f_{m-1}\|^{-4}|\varphi_m|^2. \tag{38}$$

Proof. The proof depends on the compatibility or integrability equations

$$\partial\bar{\partial} f_j = \bar{\partial}\partial f_j \tag{39}$$

satisfied by the (finite) sequence $f_1, \ldots, f_m$ generated by the superconformal harmonic map $f : M \to \mathbb{S}_1^{2m}$. We introduce functions $u_0 = 0$ and $u_j =: \log \| f_j \|$, $1 \leq j \leq m-1$, and α, β such that $f_m = \alpha F_{2m-1} - \beta F_{2m}$. Then using (7) and straightforward calculation, the integrability conditions (39) are given in terms of u_j, α, β and σ_m by the following system of PDE:

$$\begin{aligned}
&2\partial\bar{\partial} u_j = e^{2(u_{j+1}-u_j)} - e^{2(u_j - u_{j-1})}, \qquad j = 1, \ldots, m-2,\\
&2\partial\bar{\partial} u_{m-1} = (|\alpha|^2 - |\beta|^2) e^{-2u_{m-1}} - e^{2(u_{m-1}-u_{m-2})},\\
&\mathrm{Im}(\bar{\partial}\sigma_m) = -e^{-2u_{m-1}} \,\mathrm{Im}(\alpha\bar{\beta}),\\
&\bar{\partial}\alpha = \bar{\sigma}_m \beta,\\
&\bar{\partial}\beta = \bar{\sigma}_m \alpha.
\end{aligned} \tag{40}$$

Using (32) and (36) and the above system we obtain the normal curvatures in terms of u_j and α, β:

$$\begin{aligned}
&K_j = e^{-2u_1}[e^{2(u_j - u_{j-1})} - e^{2(u_{j+1}-u_j)}], \quad j = 1, \ldots, m-2,\\
&K_{m-1} = e^{-2u_1}[e^{2(u_{m-1}-u_{m-2})} - (|\alpha|^2 - |\beta|^2) e^{-2u_{m-1}}],\\
&K_m = 2e^{-2u_1} e^{-2u_{m-1}} Im(\alpha\bar{\beta}).
\end{aligned} \tag{41}$$

From which the sum of the first $m-1$ curvatures gives (37). Squaring (36) and (37) we obtain identity (38). □

Corollary 5.3. Away the zeros of the complex Hopf differential Q the normal curvatures of a superconformal harmonic map $f : M \to \mathbb{S}_1^{2m}$ satisfy the following identity:

$$\Delta_g \log\left[\left(1 - \sum_{j=1}^{m-1} K_j\right)^2 + K_m^2\right] = 4(K_1 + K_{m-1}). \tag{42}$$

Proof. Away the zeros of Q, we can take logarithm at both sides of identity (38) and apply the Laplacian Δ_g. Then since $\log |\varphi_m|^2$ is a (local) harmonic function on M, we obtain (42). □

Remark 5.3. The above identity generalizes formula (1) in Theorem (i) of the paper of Sakaki [22]. Identity (42) implies that the metric $\mathrm{d}s^2 = ((1 - \sum_{j=1}^{m-1} K_j)^2 + K_m^2)^{(1/4)} g$, ($g$ the induced metric) has curvature $K_{m-1} \cdot (1 - \sum_{j=1}^{m-1} K_j)^2 + (K_m^2)^{(1/4)}$ at points where $Q \neq 0$.

As an application of (37) we obtain the following result.

Theorem 5.4. *Let $f : M \to \mathbb{S}_1^{2m}$ be a superconformal harmonic map of a Riemann surface. Then $K_m \equiv 0$ if and only if f(M) lies fully in a (unique) nondegenerate hyperplane V. In this case*

(a) $\sum_{j=1}^{m-1} K_j - 1 \geq 0$ *if and only if the induced metric on V has signature $(2m-1, 1)$ and f is superconformal harmonic full into $\mathbb{S}_1^{2m-1}(V)$.*
(b) $\sum_{j=1}^{m-1} K_j - 1 \leq 0$ *if and only if V is space-like and f is a superconformal harmonic full into the Euclidean unit sphere $\mathbb{S}^{2m-1}(V) \subset V$.*

In both cases identity (38) implies that $\sum_{j=1}^{m-1} K_j = 1$ can occur only at the zeros of Q which are isolated.

Remark 5.4. Note from (37) that the sign of $\sum_{j=1}^{m-1} K_j - 1$ depends on the sign of $\| f_m \|^2 = |\alpha|^2 - |\beta|^2$.

Proof of Theorem 5.4. Let $f : M \to \mathbb{S}_1^{2m}$ be superconformal harmonic. If f is not full we know by Theorem 4.1 that its image $f(M)$ lies fully in a non-degenerate hyperplane $V \subset \mathbb{R}^{2m+1}$ and hence equality holds in (26) for every local complex chart on M. In terms of the functions α and β introduced before, this equality becomes $|\alpha^2 - \beta^2| = ||\alpha|^2 - |\beta|^2|$. This forces $\operatorname{Im}(\alpha\bar{\beta}) = 0$ and so, $K_m \equiv 0$ by (36).

Conversely, if $K_m \equiv 0$, the last normal bundle N^{m-1} is $\nabla_{m-1}^{\perp}$-flat and hence it is possible to choose local $\nabla_{m-1}^{\perp}$-parallel sections F_{2m-1}, F_{2m} of the last normal bundle N^{m-1} satisfying (29). Then $\sigma_m \equiv 0$ and so α, β result holomorphic by (40). Also $\sigma_m \equiv 0$ in (34) yield

$$\begin{aligned} \partial F_{2m-1} &= -\alpha e^{-2u_{m-1}} \overline{f_{m-1}}, \\ \partial F_{2m} &= -\beta e^{-2u_{m-1}} \overline{f_{m-1}}. \end{aligned} \tag{43}$$

Also from $K_m \equiv 0$ we have by (36) that $\operatorname{Im}(\alpha\bar{\beta}) \equiv 0$ which forces $|\alpha^2 - \beta^2| = ||\alpha|^2 - |\beta|^2|$, where $\varphi_m = \alpha^2 - \beta^2 \not\equiv 0$. Then either $\alpha \not\equiv 0$ and $t\alpha = \beta$ for some real number $t \neq \pm 1$, or $\alpha \equiv 0$ and $\beta \not\equiv 0$. In the first case we have $f_m = \alpha(F_{2m-1} - tF_{2m})$ and the vector $\mathbf{n} = tF_{2m-1} - F_{2m}$ is constant by (43). Then $\mathbf{n}$ is orthogonal to f_m and to the complex space-like subbundle $W = \oplus_{j=-m+1}^{m-1} L_j$ defined in equation (24), in particular $\langle f(x), \mathbf{n} \rangle = 0$ for every $x \in M$, and so $f(M)$ is fully contained in $V = \mathbf{n}^{\perp}$ which is non-degenerate since $\langle \mathbf{n}, \mathbf{n} \rangle = t^2 - 1 \neq 0$. From (37) we have

$$\sum_{j=1}^{m-1} K_j - 1 = |\alpha|^2 \langle \mathbf{n}, \mathbf{n} \rangle e^{-2(u_1 + u_{m-1})}. \tag{44}$$

Hence $\sum_{j=1}^{m-1} K_j \geq 1$ if and only if $\langle \mathbf{n}, \mathbf{n} \rangle > 0$ if and only if $f(M)$ is contained in $V = \mathbf{n}^{\perp}$ which is a $(2m-1, 1)$ hyperplane. In this case since $\langle f, f \rangle = 1$ it follows that $f : M \to \mathbb{S}_1^{2m-1}(V)$ is a full superconformal harmonic map.

On the other hand, $\sum_{j=1}^{m-1} K_j \leq 1$ if and only if $\langle \mathbf{n}, \mathbf{n} \rangle < 0$ if and only if $f(M)$ is contained in $V = \mathbf{n}^{\perp}$ which is a space-like hyperplane and $f : M \to \mathbb{S}^{2m-1}(V)$ is a full superconformal harmonic map.

If $\alpha \equiv 0$ and $\beta \not\equiv 0$ we have $f_m = -\beta F_{2m}$ and so $\mathbf{n} = F_{2m-1}$ is a constant vector. This case is then included in the case in which the image $f(M)$ is contained in a $(2m-1, 1)$ hyperplane V. □

A frame of a map $f : M \to \mathbb{S}_1^n$ is an application $F : M \to \mathrm{O}(n, 1)$ such that $\pi \circ F = f$ where $\pi : \mathrm{O}(n, 1) \to \mathbb{S}_1^n$ is the projection $F \to F_0 =$ first column of f. When $f : M \to \mathbb{S}_1^n$ is superconformal harmonic, the harmonic sequence of f can be used to construct local frames in a simple way. In fact on a local complex chart (U, z) of M define $F = (F_0, F_1, \ldots, F_{2m-1}, F_{2m})$ by (28) and (30) so that the frame F is given by

$$\begin{aligned} F_0 &= f, \\ f_j &= \frac{\|f_j\|}{\sqrt{2}}(F_{2j-1} - \mathrm{i}F_{2j}), \quad 1 \leq j \leq m-1, \\ f_m &= \alpha F_{2m-1} - \beta F_{2m}. \end{aligned} \tag{45}$$

From the orthogonality relations satisfied by f_j the real fields F_j are mutually orthogonal and of unit length, except F_{2m}. The integrability or compatibility condition

$$\partial\bar{\partial}F = \bar{\partial}\partial F$$

satisfied by f is the same as that of the harmonic sequence of f (39) which in terms of the functions $u_j = \log \|f_j\|$ is given by the system (40).

Away from the isolated zeros of the $2m$th Hopf differential of f it is possible to find a local complex coordinate (U, z) which normalizes φ_m, i.e. $\varphi_m \equiv 1$ on U (a proof of this fact is given in [9]). In terms of α and β condition $\varphi_m = \langle f_m, f_m \rangle^{\mathrm{c}} \equiv 1$ is just $\alpha^2 - \beta^2 = 1$ on U. Then ξ be a complex function defined on U such that $\alpha = \cosh \xi$ and $\beta = \sinh \xi$. Also we define new local sections F'_{2m-1}, F'_{2m} of N^{m-1} according to

$$\begin{aligned} F'_{2m-1} &= \cosh(r)F_{2m-1} + \sinh(r)F_{2m}, \\ F'_{2m} &= \sinh(r)F_{2m-1} + \cosh(r)F_{2m}, \end{aligned} \tag{46}$$

where $r = \mathrm{Re}(\xi)$. It is easily seen that $\|F'_{2m-1}\|^2 = -\|F'_{2m}\|^2 = -1$ and $\langle F'_{2m-1}, F'_{2m} \rangle = 0$. Then in this new frame we have

$$f_m = \cos(\theta)F'_{2m-1} + \mathrm{i}\sin(\theta)F'_{2m}$$

so that $\alpha' = \cos(\theta)$, $\beta' = -\mathrm{i}\sin(\theta)$, where $\theta = \mathrm{Im}(\xi)$. It then follows that $\|f_m\|^2 = \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$.

Now note that the fourth and fifth compatibility equation in system (40) imply $\bar{\partial}\theta = \mathrm{i}\sigma_m$ and hence $\partial\bar{\partial}\theta = -\mathrm{i}\bar{\partial}\sigma_m$. Also from the third equation of (40) we get $Im(\partial\sigma) = -\cos(\theta)\sin(\theta)e^{-2u_{m-1}}$ from which we deduce that θ must satisfy

$$2\partial\bar{\partial}\theta = -\sin(2\theta)e^{-2u_{m-1}}. \tag{47}$$

We have then shown that in a local coordinate chart (U, z) where $\varphi_m \equiv 1$ it is possible to find a local frame

$$F = (f, F_1, F_2, \ldots, F_{2m-2}, F'_{2m-1}, F'_{2m})$$

such that the compatibility equations of f become the following system of elliptic nonlinear partial differential equations Toda type

$$\begin{aligned} 2\partial\bar{\partial}u_j &= e^{2(u_{j+1}-u_j)} - e^{2(u_j-u_{j-1})}, \quad j = 1, \ldots, m-2, \\ 2\partial\bar{\partial}u_{m-1} &= \cos(2\theta)e^{-2u_{m-1}} - e^{2(u_{m-1}-u_{m-2})}, \\ 2\partial\bar{\partial}\theta &= -\sin(2\theta)e^{-2u_{m-1}}. \end{aligned} \tag{48}$$

Then locally away from the zeros of the $2m$th Hopf differential $Q = \varphi_m \, dz^{2m}$ the geometry of the superconformal harmonic map f is completely determined from a solution of the above system. The frame $F = (F_1, F_2, \ldots, F_{2m-2}, F'_{2m-1}, F'_{2m})$ is called a Toda frame (cf. [9]).

Now from (36) the last normal curvature K_m in terms of θ is given by

$$K_m = \sin(2\theta)e^{-2(u_1+u_{m-1})}, \tag{49}$$

also from (37) we see that

$$\sum_{j=1}^{m-1} K_j - 1 = -e^{-2(u_1+u_{m-1})}\cos(2\theta). \tag{50}$$

Thus at points where $\sum_{j=1}^{m-1} K_j \neq 1$ we have

$$\arctan\left(\frac{K_m}{\sum_{j=1}^{m-1} K_j - 1}\right) = -2\theta \tag{51}$$

Applying the Laplacian Δ_g to both sides of (51) we arrive at the following identity which generalizes formula (3.1) obtained by Alías and Palmer [1].

$$\Delta_g \arctan\left(\frac{K_m}{\sum_{j=1}^{m-1} K_j - 1}\right) = 2K_m. \tag{52}$$

As an application of (52) we obtain the following.

Lemma 5.5. *Let M be a compact connected Riemann surface and $f : M \to \mathbb{S}_1^{2m}$ a full superconformal harmonic immersion for which $\sum_{j=1}^{m-1} K_j \neq 1$ at each point of M. Then*

$$\int_M K_m \, dA = 0.$$

Proof. Integrate identity (52) respect to dA and use the divergence theorem. □

Note that under the hypothesis of the lemma, $K_m \not\equiv 0$ so that K_m is a signed function on M.

6. Congruence

An interesting question is to determine which invariants determine a superconformal harmonic map $f : M \to \mathbb{S}_1^{2m}$ up to ambient (pseudo)-isometries. We obtain the following result which is generalizes Theorem 4.1 of [7].

Theorem 6.1. *Let $f, g : M \to \mathbb{S}_1^{2m}$ be superconformal harmonic maps from a connected Riemann surface. If they induce the same metric on M and have the same $2m$th Hopf differentials, then there is an isometry Φ of $\mathbb{S}_1^{2m}$ such that $\Phi \circ f = g$.*

Proof. Assume first that f, g are full and introduce as in [7,17] (globally defined) real forms

$$\gamma_j(f) = \tau_j(f)\, \mathrm{d}z d\bar{z}, \qquad \gamma_j(g) = \tau_j(g)\, \mathrm{d}z\, \mathrm{d}\bar{z},$$

where $\tau_j(f) = \|f_{j+1}\|^2 \|f_j\|^{-2}$ and $\tau_j(g) = \|g_{j+1}\|^2 \|g_j\|^{-2}$. Since by hypothesis $f^*\langle\, ,\, \rangle = g^*\langle\, ,\, \rangle$ we have $\gamma_0(f) = \gamma_0(g)$, and $\gamma_{-1}(f) = \gamma_{-1}(g)$. Then from system (48) on a complex chart (U, z) one has

$$\partial\bar{\partial} \log \|f_j\|^2 = \tau_j - \tau_{j-1}, \quad j = 0, 1 \ldots, m-1,$$

where $\|f_j\|^2 = e^{2u_j}$. From this using finite induction we see that γ_{-1} and γ_0 determine γ_j for $j = 1, \ldots, m$, for both f and g. In particular $\tau_j(f) = \tau_j(g)$ for $j = 0, 1, \ldots, m-1$, so that $\|f_j\|^2 = \|g_j\|^2$ for $j = 0, 1, \ldots, m$. On the other hand, by hypothesis we know that

$$\begin{aligned}
&\langle f_i, f_j\rangle = \langle g_i, g_j\rangle = 0,\ 0 < |i-j| \le 2m-1,\\
&\langle f_i, \bar{f_i}\rangle = \langle g_i, \bar{g_i}\rangle = 0,\ \ i = 1, \ldots, m-1,\\
&\langle f_m, \bar{f_m}\rangle = \langle g_m, \bar{g}_m\rangle.
\end{aligned}$$

Now $\{f_j, \bar{f}_j : j = 0, \ldots, m\}$ span $\mathbb{C}_1^{2m+1}$ so that there is a matrix $A = A(z, \bar{z}) \in U(2m, 1)$ such that

$$g_j = A f_j, \quad \bar{g}_j = A \bar{f}_j, \quad j = 0, 1, \ldots, m.$$

Thus $A = \bar{A}$ and using (7) we see that $\partial A = \bar{\partial} A = 0$ and so A is a constant matrix in $\mathrm{O}(2m, 1)$ such that $Af = g$.

If f is not full, then its image is contained in a non degenerate hyperplane $V \subset \mathbb{R}_1^{2m+1}$. Assuming that V is space-like then by Theorem 5.4 we know that $\sum_{j=1}^{m-1} K_j(f) \le 1$. Now from the hypothesis and (41) it follows that $K_j(f) = K_j(g)$, $j = 1, \ldots, m-1$, therefore $\sum_{j=1}^{m-1} K_j(g) \le 1$. On the other hand respect to any local coordinate chart of M we have $\|f_m\|^2 = |\varphi_m(f)| = |\varphi_m(g)|$ and also $\|f_m\|^2 = \|g_m\|^2$. Hence $\|g_m\|^2 = |\varphi_m(g)|$ and so $K_m(g) \equiv 0$. Applying Theorem 5.4 once more, it follows that g is not full and

$g(M)$ is contained also in a space-like hyperplane V'. Writing $V = \mathbf{n}^\perp$ and $V' = \mathbf{n}'^\perp$ with $\langle \mathbf{n}, \mathbf{n}\rangle = \langle \mathbf{n}', \mathbf{n}'\rangle = -1$, there exists a matrix $A = A(z, \bar{z}) \in U(2m, 1)$ satisfying

$$g_j = Af_j, \qquad \overline{g}_j = A\overline{f}_j, \quad j = 0, 1, \ldots, m-1, \qquad \mathbf{n}' = A\mathbf{n}.$$

Proceeding as before we can check that A is real and constant so that $A \in O(2m, 1)$ and $Af = g$. If $f(M)$ were contained in a hyperplane with induced metric of signature $(2m-1, 1)$, the proof follows an analogous argument. □

7. Polar maps

According to Theorem 5.4 the image of a non-full superconformal harmonic map $f : M \to \mathbb{S}_1^{2m}$ lies fully in a non-degenerate hyperplane $V \subset \mathbb{R}_1^{2m+1}$ which may be either space-like or have signature $(2m-1, 1)$ in the induced metric. Then from Remark 4 the sequence L_j generated by f is periodic:

$$L_{2m+j} = L_j \qquad \forall j \in \mathbb{Z}. \tag{53}$$

Recall that the last line bundle of f is non-degenerate and satisfies $L_m = \overline{L}_m = L_{-m}$. These facts allow us to define the *polar map* of f as follows. Thanks to inequality (26) on any local complex chart we have $\| f_m \|^2 = \pm|\varphi_m|$ according to the signature of the metric induced on V. Then we distinguish two cases:

(i) The hyperplane V is space-like and consequently $\| f_m \|^2 = |\varphi_m|$. In particular f_m and $\sqrt{\varphi_m}$ have the same order zeros so that one can extend the vector $(f_m/\sqrt{\varphi_m})$ across its singularities by continuity (cf. [20]). It can be easily checked that it is a real vector and has square norm one. Moreover $(f_m/\sqrt{\varphi_m})$ is independent of coordinates of M. The polar map of f is then well defined by

$$f^* = \frac{f_m}{\sqrt{\varphi_m}} : M \to \mathbb{S}^{2m-1}(V) \subset V, \tag{54}$$

where $\mathbb{S}^{2m-1}(V) = \{x \in V : \langle x, x\rangle = 1\}$ is the unit sphere of V.

(ii) The induced metric on the hyperplane V has signature $(2m-1, 1)$ and so it is isometric to $\mathbb{R}_1^{2m}$. Here note that the square norm of f_m is non-positive since $\| f_m \|^2 = -|\varphi_m|$. Like in the previous case the vector $(f_m/\sqrt{\varphi_m})$ can be extended by continuity across its singularities and does not depend on local coordinates in M. However it is not a real vector since as consequence of $\overline{f_m} = -\frac{\overline{\varphi_m}}{|\varphi_m|} f_m$ we have,

$$\overline{\left(\frac{f_m}{\sqrt{\varphi_m}}\right)} = -\frac{\overline{\varphi_m} f_m}{|\varphi_m|\sqrt{\overline{\varphi_m}}} = -\frac{\sqrt{\overline{\varphi_m}} f_m}{\sqrt{\varphi_m \overline{\varphi_m}}} = -\frac{f_m}{\sqrt{\varphi_m}}.$$

In this case defining $f^* = \frac{\pm \mathrm{i} f_m}{\sqrt{\varphi_m}}$ $(i = \sqrt{-1})$, it follows that f^* is a real vector with square norm -1 lying in V which is independent on local coordinates of M. We define the polar map of *f* in this case by

$$f^* = \frac{\pm \mathrm{i} f_m}{\sqrt{\varphi_m}} : M \to \mathbb{H}^{2m-1}(V), \subset V \tag{55}$$

where the sign in (55) depends on a choice of one of the sheets of the hyperboloid $\{x \in V : \langle x, x\rangle = -1\}$ defining the hyperbolic space $\mathbb{H}^{2m-1}(V)$.

In both cases we show that f^* is a harmonic map as follows. Away of the zeros of the $2m$th Hopf differential of *f* it is possible to find a local complex chart (U, z) respect to which the $2m$th Hopf differential is normalized: $\varphi_m = 1$ on U (cf. [9]). Then if V is space like, $f^* = f_m$ on U, and using the periodicity of the harmonic sequence of *f* (53) it follows that ∂f^* is a local section of L_{-m+1}. In particular $\langle \partial f^*, \partial f^*\rangle^{\mathrm{c}} = 0$ which shows that f^* is a (weakly) conformal map. On the other hand, from Lemma 3.1 and its corollary we get the following conditions:

$$\langle \partial\overline{\partial} f^*, f_j\rangle = 0, \quad j \neq m,$$
$$\langle \partial\overline{\partial} f^*, f^*\rangle = -\langle \partial f^*, \partial f^*\rangle \ ,$$

and so $\partial\overline{\partial} f^* = -\langle \partial f^*, \partial f^*\rangle f^*$ which is the harmonic map equation for $f^* : M \to \mathbb{S}^{2m-1}(V)$. The proof in the second case (ii) follows along the same lines as in case (i) but, for $f^* : M \to \mathbb{H}^{2m-1}(V)$ the harmonic map equation is $\partial\overline{\partial} f^* = \langle \partial f^*, \partial f^*\rangle f^*$ (cf. [17]).

On the other hand, using (53) and Lemma 3.1 we compute

$$\varphi_m = \langle f_m, f_m\rangle^{\mathrm{c}} = -\langle f_{m+1}, f_{m-1}\rangle^{\mathrm{c}} = \cdots = (-1)^k \langle f_{m+k}, f_{m-k}\rangle^{\mathrm{c}} = (-1)^m \langle f_{2m}, f\rangle.$$

From which $f_{2m} = (-1)^m \varphi_m f$. Normalizing $\varphi_m \equiv 1$ respect to a local chart (U, z), we have $f_{2m} = (-1)^m f$. Then if $f(M)$ lies in a space-like hyperplane V we have $f^* = f_m$ and so $\langle f_m^*, f_m^*\rangle^{\mathrm{c}} = \langle f_{2m}, f_{2m}\rangle^{\mathrm{c}} = 1$. Moreover, $f_j^* = f_{m+j},\ j = 1, \ldots, m$ on U. Thus f^* is a full superconformal map with the same $2m$th Hopf differential as *f*.

If $f(M)$ lies in a hyperplane V of signature $(2m-1, 1)$ the situation is analogous but now $f^* = i f_m$ on U. In particular $f_j^* = i f_{m+j},\ j = 1, \ldots, m$ and $\langle f_m^*, f_m^*\rangle^{\mathrm{c}} = -1$ on U. Note that in both cases $L_j^* = L_{m+j}, \forall j \in \mathbb{Z}$ holds. We have thus proved the following.

Theorem 7.1. *Let $f : M \to \mathbb{S}_1^{2m}$ be a non-full superconformal harmonic map. If the image $f(M)$ lies in a space-like hyperplane $V \subset \mathbb{R}_1^{2m-1}$ then the polar map $f^* = (f_m/\sqrt{\varphi_m}) : M \to \mathbb{S}^{2m-1}(V)$ is a full superconformal harmonic map into the Euclidean unit sphere of V which has the same 2mth Hopf differential as f.*

If f(M) lies in a $(2m-1, 1)$-hyperplane $V \subset \mathbb{R}_1^{2m+1}$ then the polar map $f^* = (\pm \mathrm{i} f_m/\sqrt{\varphi_m}) : M \to \mathbb{H}^{2m-1}(V)$ is a full superconformal harmonic map in the sense of [17]. *In this case, the 2mth Hopf differentials of f and f^* have opposite signs.*

In [17] superconformal harmonic maps of non-compact Riemann surfaces into real hyperbolic spaces were considered and harmonic sequences were constructed. Since there are no non-constant harmonic maps of compact surfaces into real Hyperbolic spaces we obtain the following corollary.

Corollary 7.2. *There exist no non-constant superconformal harmonic map of a compact surface into odd-dimensional De Sitter space-times* $\mathbb{S}_1^{2m-1}$ *with* $m \geq 2$.

Proof. Note first that by Remark 4 f is (linearly) full. Now view $\mathbb{S}_1^{2m-1} = \mathbb{S}_1^{2m} \cap e_0^{\perp}$ and consider f as a map with target $\mathbb{S}_1^{2m}$ whose image is contained into the hyperplane $e_0^{\perp} \subset \mathbb{R}_1^{2m+1}$. Then since e_0 is space-like, $e_0^{\perp}$ has signature $(2m-1, 1)$ and hence the polar map of f takes values in the hyperbolic space inside $e_0^{\perp}$ which we denote simply by $\mathbb{H}^{2m-1}$.

Since M is compact the polar map f^* of f is constant and $0 = \langle f(x), f^*(x)\rangle = \langle f(x), f^*\rangle$ for every $x \in M$. Hence $f(M) \subset V = (f^*)^{\perp}$. This forces $f_j \in V$ for every $j \in \mathbb{Z}$ and in particular $\langle f_m, f^*\rangle = 0$. Take a complex chart (U, z) in M such that $\varphi_m = 1$ on U. Then $f^* = \mathrm{i} f_m$ and hence $0 = \langle f_m, f^*\rangle = -\mathrm{i}\|f_m\|^2 = i$ on U, which is a contradiction. □

Even the simplest case $m = 2$ in the above corollary is interesting. Recall from Section 3 that a conformal minimal immersion $f : M \to \mathbb{S}_1^3$ is superconformal if and only if its umbilic points (if any) are isolated.

Corollary 7.3. *There is no conformal minimal immersion of a compact surface M* into $\mathbb{S}_1^3$ *with isolated umbilic points.*

The reader may find interesting to compare our results with those of polar maps for harmonic maps of surfaces in Euclidean spheres which were studied in detail in the paper by Miyaoka [20]. Further applications of polar maps of space-like surfaces in De Sitter space-times will be considered in a future paper.

8. Superconformal minimal tori

Bolton and Woodward in [6] gave a new proof of a celebrated theorem by Calabi [13] characterizing isotropic minimal full immersions or the two sphere $\mathbb{S}^2$ in an Euclidean n-sphere by computing the degrees of the line bundles L_j determined by the immersion. On the other hand, Sakaki in [22] provided a characterization of minimal 2-tori in a four-dimensional Lorentz space-form using a simplified version of identity (42). Inspired by both papers we give here a characterization of superconformal minimal immersions of 2-tori in $\mathbb{S}_1^{2m}$ using essentially identity (42) combined with ideas from [6] and the machinery of harmonic sequences developed in Section 3.

We review without proofs from [6] the relationship between the harmonic sequence and higher order fundamental forms of isotropic harmonic maps into a Riemannian space-form which thanks to Lemma 3.1 can be applied to the case of superconformal harmonic maps $f : M \to \mathbb{S}_1^{2m}$.

Recall from Section 5 that the direct sum complex bundle $L_1 \oplus L_2 \oplus \cdots \oplus L_j$ identifies with the jth *osculating bundle* T^j of f whose fiber at $p \in M$ (for generic points) is spanned by the ∇-derivatives of f of order up to j at p. The higher fundamental forms of f are defined inductively as follows.

Let F^1 be the T^1-valued form defined by $F^1 = df$. For $2 \leq j \leq m$ let F^j be the T^{j-1}-valued j-form defined by

$$F^j(X_1, X_2, \ldots, X_j) = [(\nabla_{X_p} F^{j-1})(X_1, X_2, \ldots, X_{j-1})]^{(T^{j-1})^\perp},$$

where $X_1, X_2, \ldots, X_j$ are vector fields on M and ∇ is covariant differentiation determined by the connections on $(T^{j-1})^\perp$ and $\mathbb{S}_1^{2m}$. It is shown in [6,8] that F^j is a symmetric j-form and its complex extension $\tilde{F}^j$ to $(TM^{\mathbb{C}})^{\otimes j}$ satisfies

$$\tilde{F}^j(\underbrace{\partial\otimes, \ldots, \otimes\partial}_{j}) = f_j, \quad 1 \leq j \leq m,$$

where f_j is the holomorphic section of L_j. It is shown in [6] that $\tilde{F}^j = f_j\, \mathrm{d}z^j$ defines a global holomorphic section of the complex tensor product bundle

$$L_j \otimes L_1^{-(j)} := L_j \otimes \overbrace{L_1^{-1} \otimes \cdots \otimes L_1^{-1}}^{j},$$

where L_1^{-1} denotes the complex line bundle with transition functions g_{ij}^{-1}, being g_{ij} the transition functions of L_1.

Let $f : M \to \mathbb{S}_1^{2m}$ be a full superconformal minimal immersion of a compact connected Riemann surface M and assume that the immersion f has no higher-order singularities. Hence on each local domain chart (U, z), the square norms $\|f_j\|^2$ are positive for $1 \leq j \leq m-1$ by Lemma 3.1. If $(1 - \sum_{j=1}^{m-1} K_j)^2 + K_m^2 > 0$ holds on the whole of M, then (42) holds globally on M. Integrating (42) respect to the area form $\mathrm{d}A$ of the induced metric g on M and using the divergence theorem we obtain

$$0 = \int_M \Delta_g \log\left[\left(1 - \sum_{j=1}^{m-1} K_j\right)^2 + K_m^2\right] \mathrm{d}A = 4\int_M K_1\, \mathrm{d}A + 4\int_M K_{m-1}\, \mathrm{d}A.$$

Hence

$$0 = \int_M K_1\, \mathrm{d}A + \int_M K_{m-1}\, \mathrm{d}A. \tag{56}$$

By the Gauss-Bonnet Theorem the first integral above equals $4\pi(1 - \mathrm{gen}(M))$ where $\mathrm{gen}(M) = $ genus of M.

On the other hand, note that

$$\frac{1}{4\pi}\int_M K_{m-1}\, \mathrm{d}A = \int_M c_1(L_{m-1}),$$

where

$$c_1(L_{m-1}, h) = \frac{1}{2\pi \mathrm{i}} \partial\bar{\partial} \log \| f_{m-1} \|^2 \mathrm{d}z \wedge d\bar{z}$$

is the first Chern class of the complex line bundle L_{m-1} and h is a Hermitian metric on L_{m-1}. It is known that the cohomology class defined by $c_1(L_{m-1}, h)$ in $H^2(M, \mathbb{C})$ is independent of the choice of a Hermitian metric on L_{m-1} (cf. [19]). Hence it is denoted simply by $c_1(L_{m-1})$ and the degree of L_{m-1} is defined by $\deg(L_{m-1}) = \int_M c_1(L_{m-1})$.

Eq. (56) becomes

$$0 = 4\pi(1 - \mathrm{gen}(M)) + 4\pi \deg(L_{m-1}). \tag{57}$$

On the other hand, since $c_1(L_1^{-1}) = -c_1(L_1)$ we have

$$c_1(L_{m-1} \otimes (L_1^{-1})^{(m-1)}) = c_1(L_{m-1}) - (m-1)c_1(L_1). \tag{58}$$

Hence integrating this equality on M, we obtain

$$4\pi \deg(L_{m-1}) - 4\pi(m-1)(1 - \mathrm{gen}(M)) = 4\pi \deg(L_{m-1} \otimes (L_1^{-1})^{(m-1)}). \tag{59}$$

Now since our immersion f has no higher-order singularities, then $\tilde{F}_{m-1}$ is a non-vanishing globally defined holomorphic section of the complex tensor bundle $L_{m-1} \otimes (L_1^{-1})^{(m-1)}$. Therefore the sum of the orders of the zeros of $\tilde{F}_{m-1}$ is zero or, equivalently $\deg(L_{m-1} \otimes (L_1^{-1})^{(m-1)}) = 0$. Then from (59) we get

$$4\pi \deg(L_{m-1}) = 4\pi(m-1)(1 - \mathrm{gen}(M)),$$

which inserted in (57) gives

$$0 = 4\pi m(1 - \mathrm{gen}(M)),$$

which forces $gen(M) = 1$. We have thus proved

Theorem 8.1. *Let M be a compact connected Riemann surface and $f : M \to \mathbb{S}_1^{2m}$ a full superconformal minimal immersion having no higher-order singularities. If the Gaussian and normal curvatures of f satisfy $(1 - \sum_{j=1}^{m-1} K_j)^2 + K_m^2 > 0$ on M, then M is (topologically) a 2-torus.*

Conversely, if M is a 2-torus then passing to the universal covering space $\mathbb{C}$ of M it is possible to normalize $\varphi_m \equiv 1$ globally on M. Hence if the full superconformal minimal immersion $f : M \to \mathbb{S}_1^{2m}$ has no higher-order singularities, the inequality $(1 - \sum_{j=1}^{m-1} K_j)^2 + K_m^2 > 0$ holds on M as a consequence of (38).

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